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mixture of courtesy and of dissent, free from personal satire, has prevailed, as may be deemed fitting for the occasion, especially as such elements are of all others the most easy to be departed from.

On the Valuation of Life Contingencies by means of Tables of Single and Joint Lives. By CHARLES JAMES HARGREAVE, Esq., LL.D., F.R.S.

(Extracted by permission from the *Philosophical Magazine*, for Jan. 1853.)

LET $a_1, a_2 \,.\,.\, a_n$ be the reciprocals of the roots of the expression $1-s_1x+s_2x^2-\,.\,.\pm s_{n-1}x^{n-1}\mp s_nx^n$, which call ϕx. We have then $\phi x=(1-a_1x)(1-a_2x)\ldots(1-a_nx)$, from which the following series of values may be readily deduced, the Σ implying the sum of all the instances of the form placed under it, so that each expression is a symmetrical function of $a_1, a_2 \,.\,.\, a_n$.

$$\begin{array}{rl}
\phi(1)=(1-a_1)(1-a_2)\;\ldots\;(1-a_n). & \text{No. of terms, } 1. \\
-\phi'(1)=\Sigma a_1(1-a_2)\;\ldots\ldots\;(1-a_n). & \text{No. of terms, } n. \\
\frac{1}{2}\phi''(1)=\Sigma a_1a_2(1-a_3)\;\ldots\ldots\;(1-a_n). & \text{No. of terms, } n\frac{n-1}{2}. \\
-\frac{1}{2.3}\phi'''(1)=\Sigma a_1a_2a_3(1-a_4)\;\ldots\;(1-a_n). & \text{No. of terms, } n\frac{n-1}{2}\frac{n-2}{3}. \\
\vdots \qquad \vdots & \vdots \\
\pm\frac{1}{2.3\,.\,.p}\phi^{(p)}(1)=\Sigma a_1a_2a_3\,.\,.\,a_p(1-a_{p+1})\,.\,.\,(1-a_n). & \text{No. of terms,} n\frac{n-1}{2}\,.\,.\,\frac{n-p+1}{p}. \\
\vdots \qquad \vdots & \vdots \\
\pm\frac{1}{2.3\,.\,.(n-1)}\phi^{(n-1)}(1)=\Sigma a_1a_2a_3\,.\,.\,a_{n-1}\,.\,.\,(1-a_n). & \text{No. of terms, } n. \\
\mp\frac{1}{2.3\,.\,.n}\phi^{(n)}(1)=a_1a_2a_3\ldots a_n & .\,\text{No. of terms, } 1.
\end{array}$$

The sign is positive when p is even, and negative when p is odd. A single term of the pth expression contains p factors of the form a, and $n-p$ factors of the form $1-a$; and the expression itself is the sum of every term which can be so constructed.

Let there be n persons whose names are $A_1, A_2 \,.\,.\, A_n$; and let a_p denote the probability that A_p will be living at any given future date. Then by what precedes, we see that the probability that at the given date there will be exactly p out of the set living (implying that exactly $n-p$ in number are dead), is

$$\pm\frac{1}{2.3\,.\,.p}\phi^{(p)}(1).$$

Thus we have—

Probability that all are dead$=1-s_1+s_2-s_3+\ldots$,
Probability that exactly one is living$=s_1-2s_2+3s_3-4s_4+\ldots$,
Probability that exactly two are living$=s_2-3s_3+6s_4-10s_5+\ldots$;

and so on until we arrive at the probability that exactly one is dead, which is $s_{n-1}-ns_n$; and finally the probability that all are living, which is s_n.

In like manner, we observe that the expression

$$\phi(1)-\phi'(1)+\frac{1}{2}\phi''(1)-\ldots\pm\frac{1}{2.3..p}\phi^{(p)}(1)$$

represents the probability that p persons or less (including none) are living at the given date; and, omitting the first term, we have the probability that p persons or less are living, excluding the case of all dead. If $p=n$, the expression becomes $\phi(1-1)$ or $\phi(0)$ or 1, the event being certain.

By referring to the well-known process for obtaining the value of an annuity which depends upon the existence of any *status* of a contingent nature, the reader will without any difficulty at once perceive the correctness of the subjoined solution of the following problem:—

Problem.—To determine the value of an annuity during the continuance of any *status* which can be made to depend upon the life or death of any number of persons out of the set $A_1, A_2 \ldots A_n$.

Mode of Solution.—Find the probability (by the formulæ above given) that the *status* will be in existence after the lapse of any number of years; for $s_1, s_2 \ldots$ write their values $a_1+a_2+a_3+\ldots$, $a_1a_2+a_1a_3+\ldots$; and, if necessary, multiply the expression out, so as to get rid of all brackets; then for each small letter write the corresponding large letter; and the result is the value of the annuity in question expressed in values of annuities on single and joint lives, on the assumption that A_n means the value of an annuity on the life of A_n; $A_m A_n$ the value of an annuity on the joint lives of A_m and A_n; and so on for any number of persons. The most convenient notation is to consider these values as framed on the assumption that unity denotes the value of a present perpetuity; so that A_n means the ratio of the value of an annuity on the life of A_n to the value of a perpetuity, or the value of the annuity on the assumption that a perpetuity is 1. Calling A_n the value of such a *status*, the value of a perpetuity expectant on the determination or failure of the *status* is $1-A_n$. The preceding rule, therefore, determines also the value of a perpetuity expectant on the failure of any *status* formed out of n lives, or payable only when the *status* does not exist.

The above mode of solution will be rendered plain by a few examples.

Let us take every contingency that can be made from the lives of A_1 and A_2. Here

$$\phi(1)=1-s_1+s_2, \quad -\phi'(1)=s_1-2s_2, \quad \frac{1}{2}\phi''(1)=s_2,$$

which give the following probabilities and values:—

Probability that both are dead is $\phi(1)$ or $1-(a_1+a_2)+a_1a_2$;
Value of perpetuity expectant on death of survivor, $1-(A_1+A_2)+A_1A_2$;
Probability that one or both are living, $a_1+a_2-a_1a_2$;
Value of annuity for two lives and life of survivor, $A_1+A_2-A_1A_2$.

These two instances are complementary to each other.

Probability that one only is living is $-\phi'(1)$ or $a_1+a_2-2a_1a_2$;

Value of annuity for life of survivor expectant on the death of one of the two is $A_1+A_2-2A_1A_2$; and value of present perpetuity, payment of which is suspended from death of first to death of second, is $1-(A_1+A_2)+2A_1A_2$.

Probability that both are living is $\frac{1}{2}\phi''(1)$ or s_2 or a_1a_2; value of annuity on joint lives is A_1A_2; value of perpetuity expectant on death of one of the two is $1-A_1A_2$.

Now let us take the contingencies from the lives of three persons, A_1, A_2, and A_3. Here

$$\phi(1)=1-s_1+s_2-s_3; \quad -\phi'(1)=s_1-2s_2+3s_3; \quad \frac{1}{2}\phi''(1)=s_2-3s_3;$$

$$-\frac{1}{2.3}\phi'''(1)=s_3.$$

Probability that all are dead is $1-s_1+s_2-s_3$; value of perpetuity expectant on death of last survivor is

$$1-(A_1+A_2+A_3)+(A_1A_2+A_1A_3+A_2A_3)-A_1A_2A_3;$$

and, taking the complement, we have value of annuity for the lives and life of the survivor

$$A_1+A_2+A_3-(A_1A_2+A_1A_3+A_2A_3)+A_1A_2A_3.$$

Probability that one only is living is $s_1-2s_2+3s_3$; and value of annuity for life of last survivor expectant upon the decease of both the others is

$$A_1+A_2+A_3-2(A_1A_2+A_1A_3+A_2A_3)+3A_1A_2A_3.$$

Probability that one or none is living is $1-s_2+2s_3$; therefore value of perpetuity expectant on the death of two of the three is $1-(A_1A_2+A_1A_3+A_2A_3)+2A_1A_2A_3$; and value of present annuity payable so long as two or more of the three are living is

$$A_1A_2+A_1A_3+A_2A_3-2A_1A_2A_3.$$

Probability that exactly two are living is s_2-3s_3; and value of annuity for joint life of the survivors expectant upon the decease of the first of the three is

$$A_1A_2+A_1A_3+A_2A_3-3A_1A_2A_3.$$

Probability that two or one are living is s_1-s_2; so that value of annuity for life of the two survivors and the survivor of them expectant on the decease of the first of the three is

$$A_1+A_2+A_3-(A_1A_2+A_1A_3+A_2A_3).$$

Probability that all are living is s_3; so that value of annuity for the joint lives of the three is $A_1A_2A_3$; and the complementary perpetuity is $1-A_1A_2A_3$.

The above formulæ thus solve every possible contingency that can be constructed out of a single set of existing lives, treating them always as a set, and abstaining, as we have done, from dividing the set into individuals; and the results are obtained by merely handling the formulæ without the necessity of entering into any consideration of the various contingencies that may arise. Among the more obvious results are the following, which I presume are well known. The value of a perpetuity expectant upon the death of the survivor of a set of lives is

$$1-\Sigma(A)+\Sigma(AA)-\Sigma(AAA)+\ldots;$$

and the value of an annuity for their lives and the life of the survivor is

$$\Sigma(A)-\Sigma(AA)+\Sigma(AAA)-\ldots;$$

the former being, (perpetuity) + (annuities for joint lives in even combinations) — (annuities for single lives and joint lives in odd combinations); and the latter being, (annuities for single lives and joint lives in odd combinations) — (annuities for joint lives in even combinations).

Let us now introduce another set of lives B_1, $B_2 \ldots B_m$, and construct a *status* out of them which is to coexist with the *status* constructed out of the former set. We have merely to find the probability of the new *status* being in force at a given time, and multiply it by the probability of the old *status;* and when the multiplications are performed, change the small letters into large ones as before. The same process may be extended to any number of coexistent *status.*

Thus, suppose we require the value of an annuity for the lives of A_1 and A_2, and the survivor commencing at the death of B_1; we have merely to multiply $a_1+a_2-a_1a_2$ by $1-b_1$; which gives, changing the letters,

$$A_1+A_2-A_1A_2-A_1B_1-A_2B_1+A_1A_2B_1.$$

The complexity of the *status* only increases the labour of working out the results, without in any way adding to the real difficulty of the problem. Thus, suppose we require the value of an annuity for the lives of A_1 and A_2 and the survivor of them, commencing at the death of such one of the set B_1, B_2 and B_3 as shall die second, with the proviso that it shall only be payable so long as there are either exactly two or exactly three of the set C_1, C_2, C_3, and C_4 living; we have to multiply together

$$a_1+a_2-a_1a_2,$$
$$1-(b_1b_2+b_1b_3+b_2b_3)+2b_1b_2b_3,$$
$$c_1c_2+c_1c_3+c_1c_4+c_2c_3+c_2c_4+c_3c_4-2(c_1c_2c_3+c_1c_2c_4+c_1c_3c_4+c_2c_3c_4)$$
$$+2c_1c_2c_3c_4;$$

and after the multiplications, to change the letters into the corresponding names. The result would be given in this case in terms of no less than 165 annuities on joint lives.

The preceding formulæ may be adapted to the case of contingencies involving not merely lives, but also a fixed term of years, it being assumed that the term, as well as the lives, begins to run *in præsenti.* Thus, if T denote the value of an annuity for a term, and A_1T the value of an annuity for so much of the term as A_1 shall live, all the results above given remain applicable if we substitute T for one of the lives. Thus A_1+T-A_1T is the value of an annuity for the term and the life, whichever shall last the longest; A_1+T-2A_1T is the value of an annuity for the residue of one period (the term or the life) after the other has expired. So $A_1+A_2+T-(A_1A_2+A_1T+A_2T)+A_1A_2T$ is the value of an annuity for two lives and a term, whichever shall last the longest.

In these investigations A_n denotes the present value of an annuity which is payable at intervals of a year from the present time, provided the life be in existence on the day of payment; the complement of this, or $1-A_n$, must therefore be understood as denoting a perpetuity expectant on the death of A_n, on the assumption that it commences in point of interest at the beginning of the year in which the life drops, or that its first payment takes place at the beginning of the year ensuing the fall of the life. Unless this assumption be made, the two added together will not constitute a present perpetuity. In like manner, an annuity commencing at the death of A_1, and continuing until the death of A_2, is represented by $A_2 - A_1A_2$, on the assumption that it begins in interest at the commencement of the year in which A_1 dies. So, when we represent by $M(1-A_n)$ the value of a sum M payable on the death of A_n, we mean that it is payable at the commencement of the year of A_n's death; and if it be not payable until the end of that year, we must take a year's discount from it. The diminution of value occasioned by payment being postponed until the fall of a life or other *status* may be regarded as a particular mode of discounting the sum, the rate of discount depending on the nature of the *status*. Thus the value of M pounds payable at the death of the survivor of $A_1A_2 \ldots A_n$ is $M(1-\Sigma A+\Sigma(AA)-\Sigma(AAA)+\ldots)$; and the value of M pounds payable after the decease of n lives where they are not coexistent, but each life (or rather a life of that value) is nominated on the death of his predecessor (or more strictly at the beginning of the year in which he dies), is

$$M(1-A_1)(1-A_2)\ldots(1-A_n).$$

If all the lives have the same value P, the value of M pounds payable after the death of n such lives, each life being nominated at the commencement of the year in which his predecessor died, is $M(1-P)^n$; from which we at once perceive that the present value of a series of payments of M pounds, one made now, another expectant on death of P, the next expectant on the death of another nominee P, and so on *ad infinitum*, is

$$M(1+(1-P)+(1-P)^2+\ldots) \text{ or } \frac{M}{P};$$

and if such an interest be expectant on the fall of an existing life A_n, its value is $M(1-A_n)\div P$. Payments of this nature, which go by the name of fines, are in practice made payable at the end of the year in which the life falls, and the new life is not nominated until that time; this adds one year certain to the value of every life, so that $(1-A_n)$ becomes $v(1-A_n)$, and P becomes $1-(1-P)v$, v being $1\div(1+r)$, where r is the rate of interest per pound per annum. These give as the present value of such a fine $M(1-A_n)\div(P+r)$, which is in substance the same formula as that adopted by Mr. Milne.

The reader will be cautious not to confound $(1-A_1)(1-A_2)$ with $1-(A_1+A_2)+A_1A_2$; the former is the value of a perpetuity expectant on the death of two successive lives, the latter is a perpetuity on the death of the survivor of two concurrent lives of the same respective values. If the former expression be multiplied out, it should be written thus: $1-(A_1+A_2)+A_1.A_2$, so as to show that the last term is a product, and not an expression of the value of joint lives. It is evident that in dealing with successive lives of different values, it is immaterial in what order they are taken.

The formulæ at the head of this paper may be applied not merely to lives, but to any other set of events, the individual probabilities of which are known. Thus if there are n events whose respective probabilities are $a_1, a_2 \ldots a_n$, then

$$\pm\phi^{(p)}(1)\div(2.3..p)$$

is the probability that exactly p will happen; and

$$\phi(1)-\phi'(1)+\frac{1}{2}\phi''(1)-\ldots\pm\frac{1}{2.3..p}\phi^{(p)}(1)$$

is the probability that p or less will happen; and so on.

Dublin, *Dec* 18, 1852.

Note by the Author.—In order to render the investigation quite complete, it is necessary to point out a slight and obvious modification of the formulæ, in case two or more of the lives concerned are not lives of the same value, but lives of the same person.

If we have two lives of the same value (A), we may obtain in the result annuities of the form AA, AAB, &c., which present no peculiarity. If, however, it should happen that the two A's are the same person, we must substitute A for AA, and AB for AAB, and so on; since it is evident that that which appears in the form of an annuity for the joint lives of A and A, where they are really the same person, is merely an algebraical mode of expressing an annuity for the life of A. This change may cause some of the terms to disappear, a circumstance which will generally be found to indicate either that the problem has not been proposed in the simplest form, or that the simplest mode of solution has not been adopted. Thus, an annuity for the lives of A_1 and A_2 and the survivor, to commence on the failure of the joint existence of A_1 and B (the two A_1's being, not two lives of the same value, but lives of the same person), results in the form derived from multiplying $a_1+a_2-a_1a_2$ by $1-a_1b$, or from $a_1+a_2-a_1a_2-a_1a_1b-a_1a_2b+a_1a_1a_2b$. The last two terms thus cancelling each other, we obtain $A_1+A_2-A_1A_2-A_1B$, which is evidently correct.